MATH 1700: SECTION 12.1: THE LAW OF SINES

The next two sections provide us tools with which to solve triangles that are not right triangles. We begin with an example that reviews right triangle trigonometry for completeness.

EXAMPLE 1: Given a right triangle with a hypotenuse of length 7 units and one leg of length 4 units, find the length of the remaining side and the measures of the remaining angles. Give exact answers and decimal approximations (rounded to hundredths) and sketch the triangle.

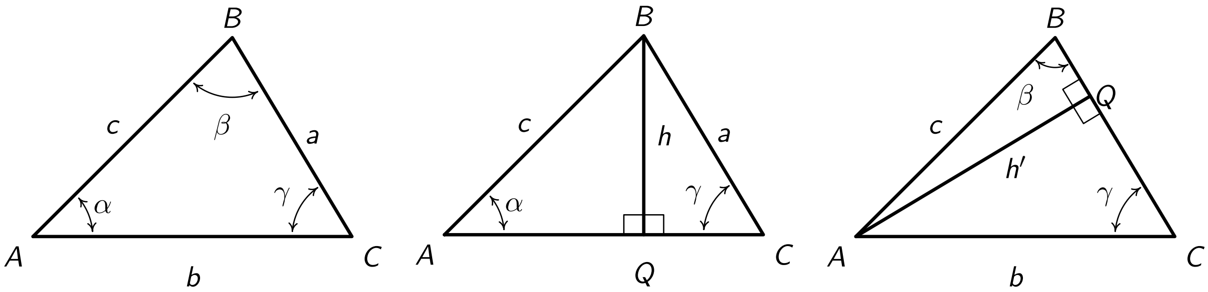
THE LAW OF SINES:

Given a triangle with angle-side opposite pairs (α, a), (β, b) and (γ, c), the following ratios hold:

$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c} \qquad \text{or, equivalently,} \qquad \frac{a}{\sin(\alpha)} = \frac{b}{\sin(\beta)} = \frac{c}{\sin(\gamma)}$$

The proof of the Law of Sines can be broken into three cases:

CASE 1: Consider the triangle $\triangle ABC$ below, all of whose angles are acute.



If we drop an altitude h from vertex B, we divide the triangle into two right triangles: $\triangle ABQ$ and $\triangle BCQ$.

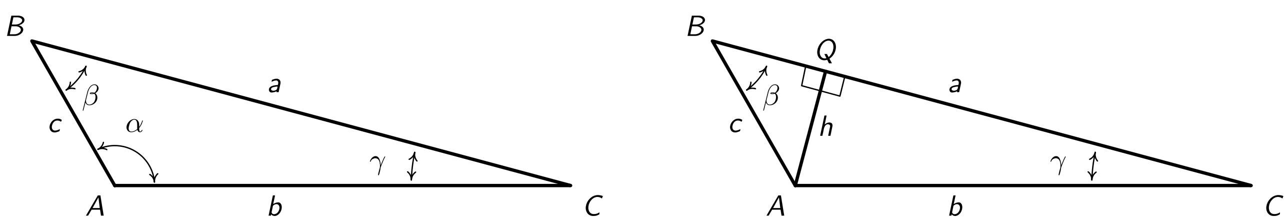
Using right triangle trigonometry, we get $\sin(\alpha) = \frac{h}{c}$ and $\sin(\gamma) = \frac{h}{a}$ so that $h = c\sin(\alpha) = a\sin(\gamma)$.

Dropping an altitude h' from vertex A also gives two right triangles and we get $h' = c\sin(\beta) = b\sin(\gamma)$.

From $h = c\sin(\alpha) = a\sin(\gamma)$, we get $\frac{\sin(\alpha)}{a} = \frac{\sin(\gamma)}{c}$ and from $h' = c\sin(\beta) = b\sin(\gamma)$, we get $\frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}$.

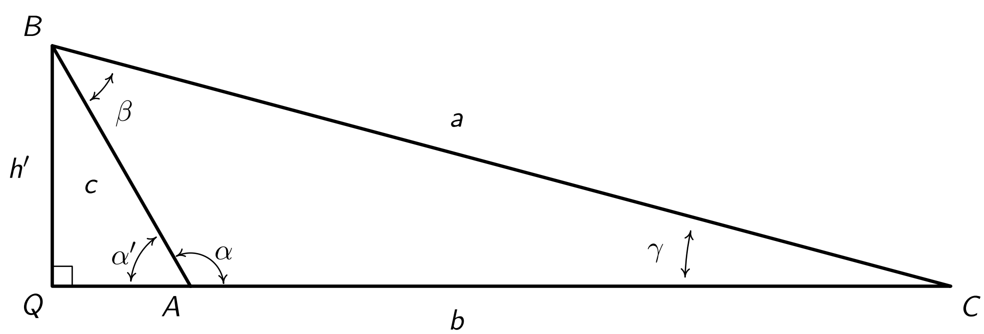
Hence, $\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}$.

CASE 2: $\triangle ABC$ below with <u>obtuse</u> angle α.



Extending an altitude from vertex A gives two right triangles like before and we get $h = b\sin(\gamma)$ and $h = c\sin(\beta)$.

Dropping an altitude from vertex B also generates two right triangles, $\triangle ABQ$ and $\triangle BCQ$.



We see $\sin(\alpha') = \frac{h'}{c}$ so that $h' = c\sin(\alpha')$. Since $\alpha' = 180^\circ - \alpha$, $\sin(\alpha') = \sin(\alpha)$, so $h' = c\sin(\alpha)$.

Using $\triangle BCQ$, we get $\sin(\gamma) = \frac{h'}{a}$ so $h' = a\sin(\gamma)$. As before, we get $\frac{\sin(\gamma)}{c} = \frac{\sin(\alpha)}{a}$, so $\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}$.

CASE 3: $\triangle ABC$ is a right triangle. In this case, the Law of Sines reduces to the definition of sine.

NOTE: In order to use the Law of Sines to solve a triangle, we need at least one angle-side opposite pair.

EXAMPLE 2: Solve the following triangles.

Give exact answers and decimal approximations (rounded to hundredths) and sketch the triangle.

1. $\alpha = 120°$, $a = 7$ units, $\beta = 45°$

2. $\alpha = 85°$, $\beta = 30°$, $c = 5.25$ units

3. $\alpha = 30°$, $a = 1$ units, $c = 4$ units

4. $\alpha = 30^\circ$, $a = 2$ units, $c = 4$ units

5. $\alpha = 30^\circ$, $a = 3$ units, $c = 4$ units

6. $\alpha = 30^\circ$, $a = 4$ units, $c = 4$ units

THE AMBIGUOUS CASE: ANGLE – SIDE – SIDE: Suppose (α, a) and (γ, c) are intended to be angle-side pairs in a triangle where α, a and c are given. Let $h = c\sin(\alpha)$

- If $a < h$, then no triangle exists which satisfies the given criteria.
- If $a = h$, then $\gamma = 90°$ so exactly one (right) triangle exists which satisfies the criteria.
- If $h < a < c$, then two distinct triangles exist which satisfy the given criteria.
- If $a \geq c$, then γ is acute and exactly one triangle exists which satisfies the given criteria

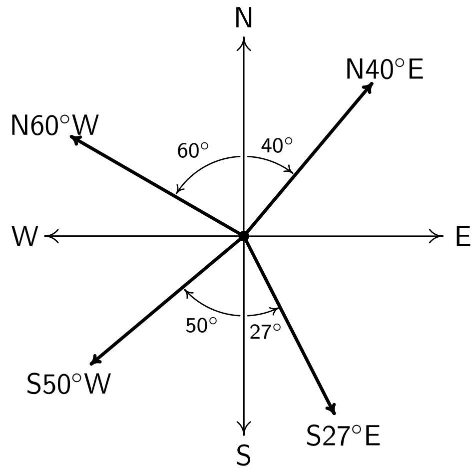
AREA OF A TRIANGLE (REPRISE): Suppose (α, a), (β, b) and (γ, c) are the angle-side opposite pairs of a triangle. Then the area A enclosed by the triangle is given by

$$A = \frac{1}{2}bc\sin(\alpha) = \frac{1}{2}ac\sin(\beta) = \frac{1}{2}ab\sin(\gamma)$$

That is, the area enclosed by the triangle $A = \frac{1}{2}$ (the product of two sides) sin(of the included angle).

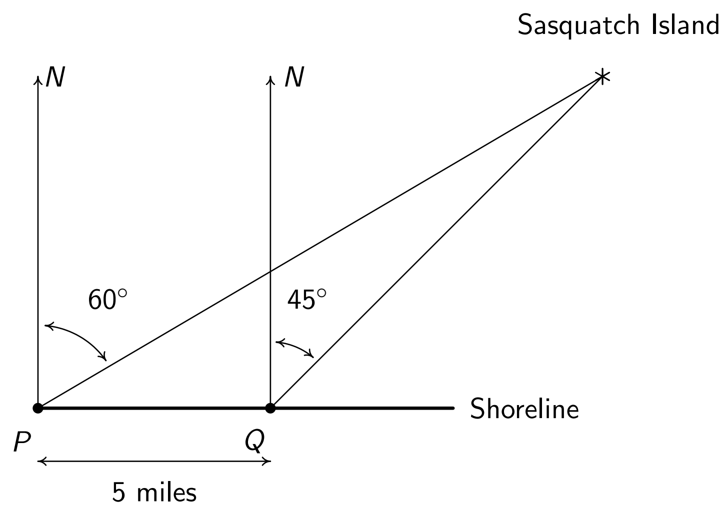
EXAMPLE 3: Find the area of the triangle given the data: $\alpha = 120°$, $a = 7$ units, $\beta = 45°$

EXAMPLES OF BEARINGS:



The cardinal directions north, south, east and west are usually not given as bearings in the fashion described above, but rather, one just refers to them as 'due north', 'due south', 'due east' and 'due west', respectively, and it is assumed that you know which quadrantal angle goes with each cardinal direction.

EXAMPLE 4: Sasquatch Island lies off the coast of Ippizuti Lake. As illustrated below, from a point P on the shore, the bearing to Sasquatch Island is observed to be N60°E. From a point Q that is 5 miles due East of P, the bearing to the island is observed to be N45°E.



Assuming the coastline continues to run due East, find the distance from the point Q to the island.

How far is the island from the coast?

HOMEWORK: Section 12.1: Page 1034: 1-21 odd, 26, 27, 33